

# Some Speed-Ups and Speed Limits for Real Algebraic Geometry

J. Maurice Rojas<sup>\*</sup>

*Department of Mathematics, City University of Hong Kong, 83 Tat Chee Avenue,  
Kowloon, HONG KONG.*

E-mail: mamrojas@math.cityu.edu.hk, Web-page: <http://www.cityu.edu.hk/ma/staff/rojas>

*Key Words:* semi-algebraic, connected components, upper bounds, fewnomials, complexity, **NP**, **BPP**, polylogarithmic

IN MEMORY OF GIAN-CARLO ROTA.

We give new positive and negative results, some conditional, on speeding up computational algebraic geometry over the reals:

1. A new and sharper upper bound on the number of connected components of a semi-algebraic set. Our bound is novel in that it is stated in terms of the volumes of certain polytopes and, for a large class of inputs, beats the best previous bounds by a factor exponential in the number of variables.
2. A new algorithm for approximating the real roots of certain sparse polynomial systems. Two features of our algorithm are (a) arithmetic complexity **polylogarithmic** in the degree of the underlying complex variety (as opposed to the super-linear dependence in earlier algorithms) and (b) a simple and efficient generalization to certain univariate exponential sums.
3. Detecting whether a real algebraic surface (given as the common zero set of some input straight-line programs) is not smooth can be done in polynomial time within the classical Turing model (resp. BSS model over  $\mathbb{C}$ ) only if  $P=NP$  (resp.  $NP \subseteq BPP$ ). The last result follows easily from an unpublished observation of Steve Smale.

## 1. INTRODUCTION AND MAIN RESULTS

We provide new speed-ups for some fundamental computations in real algebraic geometry. Our techniques are motivated by recent results from algebraic geometry but the proofs are almost completely elementary. We then conclude with a discussion of how much farther these techniques can still be pushed.

In particular, we significantly improve the best previous upper bounds on the number of connected components of a **semi-algebraic**<sup>1</sup> set, and we exhibit a new class of polynomial systems over the real numbers which can be solved within polylogarithmic time. As for complexity lower bounds, we show that if singularity detection for curves over  $\mathbb{C}$  can be done in polynomial time then, depending on the computational model, we must have  $P = NP$  or  $NP \subseteq BPP$ . This can also be thought of as a lower bound on the complexity of elimination theory, and immediately implies an analogous result on singularity detection for real algebraic surfaces.

This work is a part of an ongoing program by the author [Roj97, Roj98, Roj99b, Roj99a] to dramatically sharpen current complexity bounds from algebraic geometry in terms of more intrinsic geometric invariants. We will give precise statements of these results shortly, so let us begin by considering the number of connected components of a semi-algebraic set.

### 1.1. Sharper Intrinsic Bounds

The topology of semi-algebraic sets is intimately related to complexity theory in many ways. For example, the seminal work of Dobkin, Lipton, Steele, and Yao [DL79, SY82] (see also [BCSS98, Ch. 16]) relates upper bounds on the number of connected components to lower bounds on the algebraic circuit complexity of certain problems. More directly, upper bounds on connected components are an important ingredient in complexity upper bounds for the first order theory of the reals [BPR96].

Our first main theorem significantly improves earlier bounds on the number of connected components by Oleinik, Petrovsky, Milnor, Thom, and Basu [OP49, Mil64, Tho65, Bas96].<sup>2</sup> The main novelty of our new bound is its greater sensitivity to the monomial term structure of the input polynomials. Letting  $\mathbf{O}$  and  $\hat{e}_i$  respectively denote the origin and the  $i^{\text{th}}$  standard basis vector in  $\mathbb{R}^N$ ,  $x := (x_1, \dots, x_n)$ , and normalizing  $k$ -dimensional volume  $\text{Vol}_k(\cdot)$  so that the standard  $k$ -simplex  $\Delta_k := \{x \in \mathbb{R}^k \mid x_1, \dots, x_n \geq 0, \sum_j x_j \leq 1\}$  has volume 1, our result is the following.

MAIN THEOREM 1.1.

*Let  $f_1, \dots, f_{p+s} \in \mathbb{R}[x_1, \dots, x_n]$  and suppose  $S \subseteq \mathbb{R}^n$  is the solution set of the following collection of polynomial inequalities:*

$$f_i(x) = 0, \quad i \in \{1, \dots, p\}$$

---

<sup>1</sup>A **semi-algebraic set** is simply a subset of  $\mathbb{R}^n$  defined by the solutions of a finite collection of polynomial inequalities.

<sup>2</sup>These papers actually bound the sum of the **Betti numbers**, which in turn is an upper bound on the number of connected components. Our bounds can be extended to bound the sum of the Betti numbers as well, but this extension will be addressed in future work.

$$f_{p+i}(x) > 0, \quad i \in \{1, \dots, s\}$$

Let  $Q \subset \mathbb{R}^n$  be the convex hull of the union of  $\{\mathbf{O}, \hat{e}_1, \dots, \hat{e}_n\}$  and the set of all  $a$  with  $x^a := x_1^{a_1} \cdots x_n^{a_n}$  a monomial term of some  $f_i$ . Then  $S$  has at most

$$\min\left\{n+1, \frac{s+1}{s-1}\right\} 2^n s^n \text{Vol}_n(Q) \text{ (for } s > 0) \quad \text{or} \quad 2^{n-1} \text{Vol}_n(Q) \text{ (for } s = 0)$$

connected components.

In section 2 we show that this bound is at least as good as (and frequently much better than) the aforementioned earlier bounds. Our bound also considerably simplifies, and is competitive with, an earlier polytopal bound of Benedetti, Loeser, and Risler [BLR91, Prop. 3.6]. (We note that their polytopal bound, in addition to some minor restrictions on the  $f_i$ , applies only when  $s=0$  and  $p \leq n$ .)

It is interesting to note that there are sharper (even optimal) upper bounds relating polytope volumes and connected components for **complex** varieties, beginning with the remarkable work of Bernshtein, Kushnirenko, and Khovanski [BKK76] a bit over twenty years ago. (See also [DK86].<sup>3</sup>) However, as far as the author is aware, Main Theorem 1.1 presents the first nontrivial general upper bounds on the number of connected components of **semi-algebraic** sets with this combinatorial flavor. The work of Benedetti, Loeser, and Risler [BLR91] appears to be the first occurrence of polytopal bounds for the case where  $s=0$  and  $p \leq n$  (i.e., certain real algebraic sets).

REMARK 1.1. *Finding an **optimal** upper bound on the number of connected components of a semi-algebraic set, even in the special case of non-degenerate real algebraic sets, remains an open problem.* ■

Our bound can be further improved in various ways and this is detailed in section 3. In particular, we give sharper versions tailored for certain special cases (e.g., compact hypersurfaces and real algebraic sets), and we prove analogues (for **all** our bounds) depending only on  $n$ ,  $s$ , and the number of monomial terms which appear in at least one  $f_i$ . Khovanski appears to have been the first to consider bounds of this type for the case where  $s=0$  and  $p \leq n$  [Kho91].

The techniques involved in our proof of Main Theorem 1.1, when combined with other recent results of the author [Roj99b], also yield similar

---

<sup>3</sup> We also point out that the classical Bézout's theorem [Mum95] is optimal only for a small class of polynomial systems. So the results of [BKK76] include Bézout's theorem as a very special case.

improvements on the complexity of quantifier elimination over real-closed fields. This will be pursued in a forthcoming paper of the author.

## 1.2. Superfast Real Solving for Certain Fewnomial Systems

The complexity of solving systems of **fewnomials** (polynomials with few monomial terms<sup>4</sup>) has been addressed only recently. Indeed, the vast majority of work in computational algebra has so far been stated only in terms of degrees of polynomials, thus ignoring the finer monomial term structure. Notable exceptions include [CKS99] (solving a single univariate fewnomial over  $\mathbb{Z}$  in polynomial time), [Len98] (solving a single univariate fewnomial over  $\mathbb{Q}$  in polynomial time), and [Roj98, MP98, Roj99b, GLS99] (solving polynomial systems over  $\mathbb{R}$  or  $\mathbb{C}$  within time near polynomial in the degree of the underlying complex variety).

While it is more or less intuitively clear what it means to solve a polynomial system over  $\mathbb{Z}$  or  $\mathbb{Q}$ , let us state a motivating problem to clarify what we mean by solving over  $\mathbb{R}$ :

**PROBLEM 1.1.** *Can one  $\varepsilon$ -approximate all the roots of a univariate fewnomial of degree  $d$ , within the interval  $[0, R]$ , using significantly less than  $\Theta(d \log \log \frac{R}{\varepsilon})$  arithmetic steps? ■*

In particular, an important alternative statement is the following:

**PROBLEM 1.2.** *Can the complexity of solving fewnomials be **sub-linear** in the degree of the underlying complex variety? ■*

Finding such “super-fast” algorithms is nontrivial, even for **binomials** (i.e., quickly finding  $d^{\text{th}}$  roots) [Ye94]. The asymptotic complexity limit stated in Problem 1.1, up to a factor polylogarithmic in  $d$ , is the best current bound for solving a general univariate polynomial of degree  $d$  over  $\mathbb{C}$  [NR96]. In particular, the existence of faster algorithms for finding just the **real** roots of a degree  $d$  fewnomial was unknown until now.

Our next main theorem gives an affirmative answer to Problem 1.2, for certain fewnomial systems and univariate exponential sums over  $\mathbb{R}$ . More precisely, if  $f(x) = \sum_{a \in \mathcal{A}} c_a x^a$ , where  $\mathcal{A} \subset \mathbb{R}$  is finite and the coefficients  $c_a$  are all real, we call  $f$  a **(real) exponential  $k$ -sum**. When  $\mathcal{A} \subset \mathbb{Z}$ , we define the **degree** of such an  $f$  to be  $\max_{a, a' \in \mathcal{A}} \{a - a'\}$ . Otherwise, we set  $\mathbf{deg}(f) := \max\{a - a'\} / \min\{1, \min\{a - a'\}\}$ , where the second minimum ranges over all **distinct**<sup>5</sup>  $a, a' \in \mathcal{A}$ . We also say that  $f$  has  **$j$  sign alternations** iff there are  $j$  distinct pairs  $(a, a') \in \mathcal{A}^2$  such that

<sup>4</sup>Results on fewnomials usually hold on a much broader class of functions: the so-called **Pfaffian** functions [Kho91].

<sup>5</sup> We declare the degree of any monomial to be 0.

$c_a c_{a'} < 0$ ,  $\mathcal{A} \cap (a, a') = \emptyset$ , and  $a' > a$ . So, for instance,  $47x^{2.53} - 10.3x^{0.9} - \pi - 10x^{-3} - x^{-5.5}$  has just one sign alternation but  $x^3 - 2x + 2$  has two. Finally, when  $\mathcal{A} \subset \mathbb{Z}$ , we simply call  $f$  a ***k*-nomial**.

**MAIN THEOREM 1.2.** *Let  $f$  be any exponential  $k$ -sum of degree  $d$  with at most one sign alternation. Then, given an oracle for evaluating  $x^r$  for any  $x, r \in \mathbb{R}$ , one can  $\varepsilon$ -approximate all the roots<sup>6</sup> of  $f$  in  $(0, R)$  using  $\mathcal{O}(k(\log d + \log \log \frac{R}{\varepsilon}))$  arithmetic operations over  $\mathbb{R}$  (including oracle calls). In particular, restricting to  $k$ -nomials and removing the oracle, we can still do the same using  $\mathcal{O}(k \log d (\log d + \log \log \frac{R}{\varepsilon}))$  arithmetic operations over  $\mathbb{R}$ , with  $d$  agreeing with the usual degree of a univariate Laurent polynomial.*

We point out that even the **trinomial** case is difficult. For example, while one can count the number of real roots of a trinomial of the form  $x^d + ax + b$  within  $\mathcal{O}(\log d)$  arithmetic operations [Ric93] (regardless of sign alternations), doing the same for general trinomials was an open problem until recently [RY99]. Also, even from a numerical point of view, the use of Newton's method is subtle for trinomials: It is known that deciding whether a given initial point converges to a root of  $x^3 - 2x + 2$  is undecidable in the BSS model over  $\mathbb{R}$  (see [BCSS98, Sec. 2.4] and [Bar56]). Nevertheless, this need not stop us from finding **some** good starting point, as we will soon see.

Our algorithm, aside from an algebraic trick, closely follows an algorithm of Ye [Ye94] (for a particular class of analytic functions) which efficiently blends binary search and Newton's method. By combining these ideas with a few facts on the **Smith normal form** of an integral matrix [Ili89], we can also derive the following complexity result on binomial systems.

**MAIN THEOREM 1.3.** *Let  $c_1, \dots, c_n \in \mathbb{R} \setminus \{0\}$  and let  $[d_{ij}]$  be any  $n \times n$  matrix with nonnegative integer entries. Finally, let  $f_i := x_1^{d_{i1}} \dots x_n^{d_{in}} + c_i$  for all  $i$ . Then we can  $\varepsilon$ -approximate all the roots of  $f_1 = \dots = f_n = 0$  in the **orthant wedge**  $\{x \in \mathbb{R}^n \mid x_1, \dots, x_n \geq 0, \sum_i x_i^2 \leq R^2\}$  within*

$$\mathcal{O}((n + \log \max |d_{ij}|)^{6.376}) \text{ bit operations,}$$

*followed by*

$$\mathcal{O}\left(\log |\det[d_{ij}]| \left[ n^3 \log^2(n \max |d_{ij}|) + \log \log \frac{R}{\varepsilon} \right]\right)$$

*rational operations over  $\mathbb{R}$ .*

If the above binomial system has only finitely many **complex** roots, then their number is exactly  $|\det[d_{ij}]|$ . This follows easily from **Bernshtein's**

---

<sup>6</sup>...and of course count their number

**theorem** [BKK76]. It is also interesting to note that the fastest previous general (sequential) algorithms for polynomial system solving over  $\mathbb{R}$  or  $\mathbb{C}$ , when applied to binomial systems, run in time **polynomial** in  $|\det[d_{ij}]|$  [MP98, Roj99b, GLS99] — that is, super-linear in the degree of the underlying complex variety.

One can of course solve slightly more general systems of fewnomials by threading together the algorithms of Main Theorems 1.2 and 1.3. We will say more on the likelihood of farther-reaching extensions of our last two results after first discussing a result relating complexity classes and singularities.

REMARK 1.2. *Finding  $\varepsilon$ -approximations of roots within a suitable region is far from the strongest notion of solving a polynomial system. In particular, the spacing between roots, which of course dictates the  $\varepsilon$  one should choose, must be taken into account. A more complete and elegant framework would be to include the **condition number** [BCSS98] of the input fewnomial system in all complexity bounds. It is thus the author's intent that the preceding fewnomial complexity bounds be interpreted as a first step in this direction. ■*

### 1.3. Obstructions to Superfast Degeneracy Detection

The preceding two algorithmic results circumvent degeneracy problems in simple but subtle ways. For instance, Main Theorem 1.2 clearly deals with equations having at most one positive real root, while the binomial systems of Main Theorem 1.3 are easily seen to have no repeated complex roots (cf. section 4). Thus, the respective hypotheses of these results (restricting sign alternations and/or number of monomial terms) allow us to approximate roots without stopping for a singularity check.

It seems hard to completely solve a system of equations without knowing something about its degeneracies, either a priori or during run-time. So let us present a result which gives solid evidence that detecting degeneracies may be quite difficult. In what follows, unless otherwise mentioned, we use the standard **sparse encoding** for multivariate polynomials [Pla84, Koi96]. Thus the **size** of a polynomial like  $x^d + x - 47$  will be  $\Theta(\log d)$  and not  $\Theta(d)$ , whether in the Turing model or the BSS model over  $\mathbb{C}$ .

MAIN THEOREM 1.4. *Suppose any of the following problems can be solved in polynomial time via a Turing machine (resp. BSS machine over  $\mathbb{C}$ ). Then  $\mathbf{P} = \mathbf{NP}$  (resp.  $\mathbf{NP} \subseteq \mathbf{BPP}$ ).*

1. *Decide if an input polynomial  $f \in \mathbb{Z}[x_1]$  (resp.  $f \in \mathbb{C}[x_1]$ ) vanishes at a  $d^{\text{th}}$  root of unity, where  $d = \deg(f)$ .*

2. Decide if two input polynomials  $f, g \in \mathbb{Z}[x_1]$  (resp.  $f, g \in \mathbb{C}[x_1]$ ) have a common root.
3. Given a nonzero input polynomial  $f \in \mathbb{Z}[x_1, x_2]$  (resp.  $f \in \mathbb{C}[x_1, x_2]$ ) decide if the curve  $\{(x_1, x_2) \in (\mathbb{C}^*)^2 \mid f(x_1, x_2) = 0\}$  has a singularity.
4. Given input polynomials  $f, g \in \mathbb{Z}[x_1, x_2, x_3, x_4]$  (resp.  $f, g \in \mathbb{R}[x_1, x_2, x_3, x_4]$ ), in the straight-line program encoding, defining a surface  $S \subset \mathbb{R}^4$ , decide if  $S$  has a singularity.
5. Given any finite subset  $\mathcal{A} \subset \mathbb{Z}^2$  and a vector of coefficients  $(c_a \mid a \in \mathcal{A}) \in \mathbb{Z}^{\#\mathcal{A}}$  (resp.  $\in \mathbb{C}^{\#\mathcal{A}}$ ), decide if the  $\mathcal{A}$ -discriminant of the bivariate polynomial  $\sum_{a \in \mathcal{A}} c_a x^a$  vanishes.

REMARK 1.3. Note that in problem (4) we are already given that  $S$  is a surface. Determining whether this is true or not turns out to be **NP**-hard (resp. **NP** $_{\mathbb{R}}$ -complete) in the Turing model (resp. BSS model over  $\mathbb{R}$ ) [Koi99]. ■

For any  $\mathcal{A} \subset \mathbb{Z}^n$ , the  **$\mathcal{A}$ -discriminant**,  $\mathcal{D}_{\mathcal{A}}$ , is defined to be the unique (up to sign) irreducible polynomial in  $\mathbb{Z}[c_a \mid a \in \mathcal{A}]$  such that  $f_{\mathcal{A}}(x) := \sum_{a \in \mathcal{A}} c_a x^a$  has a singularity in its zero set (in  $(\mathbb{C}^*)^n$ )  $\implies \mathcal{D}_{\mathcal{A}} = 0$  [GKZ94]. This important operator lies at the heart of **sparse elimination theory**, which is the part of algebraic geometry surrounding this paper.

The  $\mathcal{A}$ -discriminant in fact contains all known multivariate resultants and discriminants as special cases, and also appears in residue theory and hypergeometric functions [GKZ94]. Thus, a corollary of our last main result is that sparse elimination theory, even in low dimensions, might lie beyond the reach of **P**.

REMARK 1.4. It is interesting to note that nontrivial lower bounds on the complexity of computing  $\mathcal{A}$ -discriminants in the **one**-dimensional case  $\mathcal{A} \subset \mathbb{Z}$  are unknown. However, it is easy to show (via [GKZ94, pg. 274]) that one can at least find  $\mathcal{D}_{\mathcal{A}}$  in **polynomial time** when  $\mathcal{A} \subset \mathbb{Z}^n$  has less than  $n + 3$  elements. ■

We will prove our main theorems in order of appearance, but first let us return to our study of semi-algebraic sets to see some examples.

## 2. COMPARING UPPER BOUNDS ON THE NUMBER OF CONNECTED COMPONENTS

Here we briefly compare our first main theorem to earlier bounds on the number of connected components of a semi-algebraic set.

In summary, we can compare our new bound to earlier bounds (stated in terms of total degree) in very simple polyhedral terms: Let  $\Delta_Q$  denote the smallest scaled standard  $n$ -simplex,  $\gamma\Delta_n$ , containing  $Q$ . Then, since volume is monotonic under containment, our bounds are least favorable when  $Q = \Delta_Q$ . However, in practice it will frequently be the case that  $Q$  has much smaller volume than  $\Delta_Q$ , thus accounting for improvements as good as a factor exponential in  $n$ .

### 2.1. At Least One Inequality

Assume  $s > 0$  temporarily. Letting  $d$  be the maximum of the total degrees of the  $f_i$ , the best previous general upper bounds, quoted from [BCSS98, Ch. 16, Prop. 5] and [Bas96] respectively, were  $(sd + 1)(2sd + 1)^n$  and  $(p + s)^n \mathcal{O}(d)^n$ . (The first bound is an improved version of a bound due to Milnor, Oleinik, Petovsky, and Thom [OP49, Mil64, Tho65].) Our bound is no worse than  $\min\{n + 1, \frac{s+1}{s-1}\}(2sd)^n$  (better than both preceding bounds) and is frequently much better. Consider the following examples:

**EXAMPLE 2.1. (Spikes)** Suppose we pick all the  $f_i$  to have the same monomial term structure, and in such a way that  $Q$  has small volume but great length some chosen direction. In particular, let us assume that the only monomial terms occurring in the  $f_i$  are  $1, x_1, \dots, x_{n-1}$  and  $(x_1 \cdots x_n), (x_1 \cdots x_n)^2, \dots, (x_1 \cdots x_n)^D$ . Then it is easy to check that  $Q$  is a “long and skinny” bipyramid, with one apex at the origin and the other at  $(D, \dots, D) \in \mathbb{R}^n$ . We then obtain, via two simple determinants, that  $\text{Vol}_n(Q) = D + 1$  and thus our bound reduces to  $\min\{n + 1, \frac{s+1}{s-1}\}2^n s^n (D + 1)$ . However, the aforementioned older bounds are easily seen to reduce to  $(nsD + 1)(2nsD + 1)^n$  and  $((p + s)\mathcal{O}(nD))^n$ . ■

**EXAMPLE 2.2. (Bounded Multidegree)** Suppose now that instead of bounding the total degree of the  $f_i$ , we only require that the degree of  $f_i$  with respect to any  $x_j$  be at most  $d'$ . It is then easy to check that  $Q$  is an axes parallel hypercube with side length  $d'$ . So our new bound reduces to  $\min\{n + 1, \frac{s+1}{s-1}\}(2sd')^n$ . However, the old bounds are easily seen to reduce to  $(snd' + 1)(2snd' + 1)^n$  and  $((p + s)\mathcal{O}(nd'))^n$ . ■

### 2.2. Real Algebraic Sets

Assume now that  $s = 0$ . Then the aforementioned earlier upper bounds respectively reduce to  $d(2d - 1)^n$  and  $(p\mathcal{O}(d))^n$ . Specializing Main Theorem 1.1, we obtain a bound which is no worse than  $2^{n-1}d^n$  (negligibly worse than the first, better than the second), and is frequently much better. This can



easily be seen by reconsidering our last two examples in the case  $s=0$ . (We leave this as an exercise.)

However, let us now make a fairer comparison to another polytopal bound — that of Benedetti, Loeser, and Risler [BLR91, Prop. 3.6].

REMARK 2.5. *The bound [BLR91, Prop. 3.6] was published with several typographical errors. Following inquiries from the author, Francois Loeser kindly responded via three e-mails with the following corrections:<sup>7</sup> in the notation of their bound, a hypothesis of  $k \leq n$  was missing. Also, in part (a) of their statement, the quantity  $\Phi(\Delta)$  should be replaced by  $\theta_k^n(\Delta)$ , and the last sum should be replaced by the main quantity from Prop. 3.1. Finally, in part (c), all  $j$ 's should be capitalized, and  $\theta$  should be replaced by  $\theta_{k-\#J}^n$ .* ■

The bound [BLR91, Prop. 3.6] has a recursive definition based on **mixed** volumes [GK94, DGH98]. For the sake of brevity, we will focus on the four examples given in [BLR91].

EXAMPLE 2.3. **(Four Examples from [BLR91])** *Examples (A), (B), (C), and (D) of [BLR91, Sec. 4] concern polynomial systems of the following shape: (A)  $c_0 + c_1x^a + c_2y^b$  (one polynomial, two variables), (B)  $c_0 + c_1x_1^{a_1} + \dots + c_nx_n^{a_n}$  (one polynomial,  $n$  variables), (C)  $c_0 + c_1x + c_2y + c_3(xy)^a$  (one polynomial, two variables), and (D)  $(c_0 + c_1x^a + c_2y^b, c_3 + c_4x^b + c_5y^b + c_6(xy)^b)$  (two polynomials, two variables), where the  $c_i$  are real constants and  $a, b \in \mathbb{N}$ .*

*The polytopal bound of [BLR91], when applied to these examples in the above order, respectively evaluates to  $2ab+4$ ,  $2a_1 \dots a_n + \text{Lower Order Terms}$ ,  $8a$ , and  $2ab - b^2 + \text{Lower Order Terms}$ . None of the preceding lower order terms is stated explicitly in [BLR91], and it appears that the last value is incorrect. However, a closer examination of their (corrected) bound respectively yields  $2ab + 4$ ,  $2(a_1 + 2) \dots (a_n + 2)$ ,  $8a$ , and  $8b^2 + 6ab + 8$ .*

*Main Theorem 1.1 is easily seen to respectively evaluate to  $2ab$ ,  $2^{n-1}a_1 \dots a_n$ ,  $4a$ , and  $4ab$  for these examples.* ■

More generally, it is not hard to check that our bound is usually better than that of [BLR91] when  $n$  is small or  $p$  is close to  $n$ . (Indeed, the bound of [BLR91] does not cover the case  $p > n$ .) However, the bound from [BLR91] usually wins when  $p$  is a small constant and  $n$  is large. The author hopes to combine the techniques here with those of [BLR91] in future work.

---

<sup>7</sup>Professor Loeser states that these corrections were also checked with Jean-Jacques Risler, one of the other authors of [BLR91].

### 3. PROVING MAIN THEOREM 1.1

We will first prove a sharper version of Main Theorem 1.1 for compact hypersurfaces, and then successively generalize to the case of real algebraic and semi-algebraic sets. Along the way, we give analogues of our upper bounds depending only on  $n$ ,  $s$ , and the number of monomial terms.

REMARK 3.6. *Throughout this section, “nonsingular” (or “smooth”) for a real algebraic variety will mean that the underlying **complex** variety is nonsingular in the sense of the usual Jacobian criterion (see, e.g., [Mum95]).* ■

#### 3.1. Point-Free Compact Zero Sets of a Single Polynomial

We begin with the following important special case of Main Theorem 1.1. This lemma is also frequently significantly sharper than many earlier results and may be of independent interest.

LEMMA 3.1. *Following the notation of Main Theorem 1.1, suppose  $p=1$ ,  $s=0$ , and  $S$  is compact but has no zero-dimensional components. Then  $S$  has at most  $\frac{1}{\min\{2,n\}}\text{Vol}_n(Q')$  connected components, where  $Q'$  is the convex hull of the union of  $\{\mathbf{O}\}$  and the set of all  $a$  with  $x^a$  a monomial term of  $f_1$ .*

**Proof:** The main idea will be to show that (for  $n \geq 2$ ) the number of connected components is bounded above by half the number of critical points of a projection of a perturbed version of  $S$ . This idea is quite old, but we will introduce an unusual projection which permits a much sharper upper bound than before. The case  $n=1$  of our bound is trivial, so let us assume  $n \geq 2$  henceforth.

Consider  $\tilde{f} := f_1 + \delta$ , for some  $\delta \in \mathbb{R}$  to be selected later. By Sard’s theorem [Hir94], there is a set  $W \subseteq \mathbb{R}$  of full measure such that  $\delta \in W \implies S_\delta = \{x \in \mathbb{R}^n \mid \tilde{f} = 0\}$  is nonsingular (and a hypersurface). Also, via a simple homotopy argument,  $S$  and  $S_\delta$  are both compact and have the same number of connected components, for  $|\delta|$  sufficiently small. (Much stronger versions of this fact can be found in [Bas96].) Furthermore, note that for all but finitely many  $\delta$ , no connected component of  $S_\delta$  lies inside the union of the coordinate hyperplanes. We will pick  $\delta \neq 0$  so that all these conditions, and one more to be described below, hold.

Now consider the function  $x^a$ , with  $a \in \mathbb{Z}^n \setminus \{\mathbf{O}\}$  to be selected later. Clearly, any connected component of  $S$  (not lying in a hypersurface of the form  $x^a = \text{constant}$ ) must have at least two special points: one locally maximizing, and the other locally minimizing,  $x^a$ . Since there are only finitely many connected components (by any earlier bound, e.g., [OP49]), and every component contains a curve, there must therefore be an  $a \in$

$\mathbb{Z}^n \setminus \{\mathbf{O}\}$  so that every component (not lying entirely within the union of coordinate hyperplanes) contributes at least two critical points of  $x^a$ . Pick  $a$  in this way, subject to the additional minor restriction that the g.c.d. of the coordinates of  $a$  is 1.

Note that the critical points of the function  $x^a$  on  $S_\delta$  are just the solutions in  $\mathbb{R}^n$  of

$$(\star) \quad \tilde{f} = \frac{\partial \tilde{f}}{\partial y_2} = \cdots = \frac{\partial \tilde{f}}{\partial y_n} = 0,$$

where the  $y_i$  are new variables to be described shortly. Our final condition on  $\delta$  (which is easily seen to hold for all but finitely many  $\delta$ ) will simply be that all real solutions to the above polynomial system lie in  $(\mathbb{R}^*)^n := (\mathbb{R} \setminus \{0\})^n$ . Note also that a corollary of all our assumptions so far is that the number of **complex** solutions of  $(\star)$  is finite. (This follows immediately from Sard's theorem, and the fact that the complex solutions of  $(\star)$  form an algebraic set.)

We are now essentially done: The number of connected components of  $S$  and  $S_\delta$  are the same, and the latter quantity is bounded above by half the number of critical points (on  $S_\delta$ ) of the function  $x^a$ . This number of critical points can be computed in terms of polytope volumes as follows: Via the Smith normal form [Smi61], we can find an invertible change of variables on  $(\mathbb{R}^*)^n$  such that  $y_1 := x^a$  and  $y_2, \dots, y_n$  are monomials in the  $x_i$ . Furthermore, this change of variables induces the action of a unimodular matrix on the exponent vectors of  $\tilde{f}$ . In particular,  $\tilde{f}$  can be considered as a polynomial in  $\mathbb{R}[y_1^{\pm 1}, \dots, y_n^{\pm 1}]$  and the number of monomial terms (and Newton polytope volume) of  $\tilde{f}$  is preserved under this change of variables. Thus, up to a monomial change of variables, the critical points of the function  $x^a$  on  $S_\delta$  are exactly the solutions in  $(\mathbb{R}^*)^n$  of  $(\star)$ .

The key to our new bound is to finish things off by picking a bound **other** than Bézout's theorem here. In particular, by Bernshtein's theorem [BKK76], the number of solutions in  $(\mathbb{C}^*)^n$  is at most the **mixed volume** of  $Q'$  and  $n - 1$  other polytopes with translates contained in  $Q'$ . By the monotonicity of the mixed volume [BZ88], the latter quantity is at most the mixed volume of  $n$  copies of  $Q'$  and, by the definition of mixed volume, this is just  $\text{Vol}_n(Q')$ . ■

We point out that a key ingredient in our proof is that the monomial change of variables we use (as opposed to the linear changes of variables used in most earlier treatments) preserves sparsity. This allows us to take full advantage of more powerful and refined techniques to bound the number of real roots, and thus get new bounds on the number of real connected components. For example, substituting Bernshtein's theorem for Bézout's theorem in the older proofs would not have yielded any significant improvement.

However, we need not have been so heavy-handed and only used tools over  $\mathbb{C}$ . We could have also used the following alternative bound on the number of real roots.

KHOVANSKI'S THEOREM ON REAL FEWNOMIALS 3..

**(Special Case)** [Kho91, Sec. 3.12, Cor. 6] Suppose that for all  $i \in \{1, \dots, n\}$ ,  $f_i \in \mathbb{R}[x_1, \dots, x_n, m_1, \dots, m_k]$  has total degree  $q_i$ , where the  $m_j$  are monomials in  $x$ . Let  $k$  denote the number of monomial terms which appear in at least one of  $f_1, \dots, f_n$ . Assume further that the variety  $S$  defined by  $f_1, \dots, f_n$  is zero-dimensional and nonsingular. Then  $S$  has at most  $(1 + \sum_i q_i)^k 2^{k(k-1)/2} \prod q_i$  connected components in the positive orthant. ■

We call any set of the form  $\{x \in \mathbb{R}^n \mid \pm x_1, \dots, \pm x_n \geq 0\}$  a **closed orthant**. When all signs are positive we call the corresponding closed orthant the **nonnegative orthant**. The analogous constructions where all inequalities are strict are, respectively, an **open orthant** and the **positive orthant**.

As an immediate corollary, our proof above yields the following alternative upper bound on the number of components of a smooth compact real algebraic hypersurface.

**COROLLARY 3.1.** *Following the notation of lemma 3.1, assume further that  $S$  is a smooth compact hypersurface. Then the number of connected components of  $S$  is at most  $2^{n-1}(n+1)^{k+1}2^{k(k+1)/2}$ . In particular,  $S$  has at most  $\frac{1}{2}(n+1)^k 2^{k(k-1)/2}$  connected components contained entirely within the positive orthant.*

**Proof:** Following the notation of our last proof, note that multiplying any equation of  $(\star)$  by a monomial in  $y_1, \dots, y_n$  does not affect the roots in  $(\mathbb{R}^*)^n$ . Thus, we can assume  $(\star)$  has only  $k+1$  distinct monomial terms. Also note that the monomial change of variables  $x \mapsto y$  maps orthants onto orthants, and that the case  $n=1$  is trivial. The first portion of our corollary then follows immediately from our last proof (using Khovanski's Theorem on Fewnomials with  $q_1 = \dots = q_n = 1$  instead of Bernstein's Theorem), upon counting roots in all open orthants. The second portion follows even more easily, upon observing that we do not need  $\delta$  if we only want to count critical points in an open orthant. ■

### 3.2. The Case of Real Algebraic Varieties

The next step in proving Main Theorem 1.1 is to increase the number of polynomials allowed and drop the compactness hypothesis. Again, the following result is frequently much sharper than many earlier bounds and may also be of independent interest.

LEMMA 3.2. *Following the notation of Main Theorem 1.1, suppose now that  $s=0$ , so that  $S$  is a real algebraic variety, not necessarily smooth or compact. Then  $S$  has at most  $2^{n-1}\text{Vol}_n(Q)$  connected components.*

**Proof:** The main trick is to reduce to the case considered by our preceding lemma. In particular, define  $F_{\delta,\varepsilon} := f_1^2 + \cdots + f_p^2 + \varepsilon^2(\sum_i x_i^2) - \delta^2 \in \mathbb{R}[x_1, \dots, x_n]$  and let  $S_{\delta,\varepsilon}$  be the set of real zeroes of  $F_{\delta,\varepsilon}$ . It then follows that for sufficiently small (and suitably restricted)  $\delta, \varepsilon > 0$ ,  $S_{\delta,\varepsilon}$  is a smooth compact hypersurface and the number of connected components of  $S_{\delta,\varepsilon}$  is no smaller than the number of connected components of  $S$ . The proof of this fact is standard and a very clear account can be found in [BCSS98, Sec. 16.1].

In any event, the number of connected components of  $S_{\delta,\varepsilon}$  is clearly at most  $\frac{1}{2}\text{Vol}_n(\text{Conv}(2Q' \cup \{2\hat{e}_1, \dots, 2\hat{e}_n\}))$ , thanks to our preceding lemma. Since the last quantity is just  $\frac{1}{2} \cdot 2^n \text{Vol}_n(Q)$  we are done. ■

We can combine the proof of lemma 3.2 with Khovanski's Theorem on Fewnomials to obtain the following generalization of corollary 3.1. This result, while giving a slightly looser bound than an earlier result of Khovanski [Kho91, Sec. 3.14, Cor. 5], removes all the nondegeneracy assumptions from his result.

COROLLARY 3.2. *Following the notation and assumptions of lemma 3.2, the number of connected components of  $S$  is also bounded above by  $4^{n-\frac{1}{2}}(2n+1)^{k+1}2^{k(k+1)/2}$ .*

**Proof:** Combining the proofs of lemmata 3.2 and 3.1, and since we are only counting roots in  $(\mathbb{R}^*)^n$ , we see that the number of connected components is at most half the number of solutions in  $(\mathbb{R}^*)^n$  of the following polynomial system:

$$(\star\star) \quad \bar{F}_{\delta,\varepsilon} = y_2 \frac{\partial \bar{F}_{\delta,\varepsilon}}{\partial y_2} = \cdots = y_n \frac{\partial \bar{F}_{\delta,\varepsilon}}{\partial y_n} = 0,$$

where  $\bar{F}_{\delta,\varepsilon}$  is the variant of  $F_{\delta,\varepsilon}$  where we substitute  $\sum_i y_i^2$  for  $\sum_i x_i^2$ . (It is a simple exercise to verify that the proof of lemma 3.2 still goes through with this variation.) Now simply note, via the chain rule of calculus, that every polynomial in  $(\star\star)$  is of degree at most 2 in  $y_1, \dots, y_n$  and the set of monomials appearing in  $f_1, \dots, f_p$ . Also note that the polynomials in  $(\star\star)$  are polynomials in a total of  $k+1$  monomial terms. So by Khovanski's Theorem on Real Fewnomials, and counting roots in all open orthants, we are done. ■

### 3.3. Extending to Semi-Algebraic Sets

We are now ready to prove Main Theorem 1.1.

**Proof of Main Theorem 1.1:** We reduce again, this time to lemma 3.2. The trick here is to note that every connected component of  $S$  is in turn a connected component of  $S'$  where  $S' := \{x \in \mathbb{R}^n \mid f_1(x) = \dots = f_p(x) = 0, f_{p+1}(x) \neq 0, \dots, f_{p+s}(x) \neq 0\}$ . Every connected component of  $S'$  is in turn a projection (onto the first  $n$  coordinates) of a connected component of  $S''$ , where  $S'' \subset \mathbb{R}^{n+1}$  is the real zero set of the polynomial system  $(f_1, \dots, f_p, -1 + z \prod_{i=p+1}^{p+s} f_i)$ . This reduction is not new and appears, among other places, in [BCSS98, Sec. 16.3].

Now lemma 3.2 tells us that the number of connected components of  $S''$  is at most  $2^n$  times the  $(n+1)$ -dimensional volume of  $\text{Conv}(P_1 \cup (P_2 \times \hat{e}_{n+1}))$ , where  $P_1$  (resp.  $P_2$ ) is the union of  $\{\mathbf{O}, \hat{e}_1, \dots, \hat{e}_n\}$  and the Newton polytopes of  $f_1, \dots, f_p$  (resp. the **Minkowski sum** of the Newton polytopes of  $f_{p+1}, \dots, f_{p+s}$ ). However, it is a simple exercise to show that  $P_2 \subseteq P_3$  where  $P_3$  is the union of  $\{\mathbf{O}, \hat{e}_1, \dots, \hat{e}_n\}$  and the Newton polytopes of  $f_{p+1}, \dots, f_{p+s}$ , scaled by a factor of  $s$ . Now note that  $P_2 \subseteq Q$ ,  $P_3 \subseteq sQ$  and  $\text{Conv}(P_1 \cup (P_2 \times \hat{e}_{n+1})) \subseteq \text{Conv}(Q \cup (sQ \times \hat{e}_{n+1}))$ .

If  $s > 1$  then the last polytope is in turn contained in a pyramid  $P$  with apex at  $(0, \dots, 0, \frac{-1}{s-1})$  and base  $Q \times \hat{e}_{n+1}$ . So we obtain that the number of connected components of  $S$  is at most  $2^n \text{Vol}_{n+1}(P) = 2^n \frac{s+1}{s-1} \text{Vol}_n(sQ) = \frac{s+1}{s-1} 2^n s^n \text{Vol}_n(Q)$ .

If  $s = 1$  then  $\text{Conv}(Q \cup (sQ \times \hat{e}_{n+1})) = [\mathbf{O}, \hat{e}_{n+1}] \times Q$ . So, similar to the previous case, the number of connected components of  $S$  is at most  $2^n \text{Vol}_{n+1}(P) = 2^n n \text{Vol}_n(Q)$ .

Now note that the number of connected components of  $S$  will always be at most  $\min\{n+1, \frac{s+1}{s-1}\} 2^n s^n \text{Vol}_n(Q)$ , with the possible exception of the case  $(n, s) = (1, 2)$ . So we need only check this final case. However, this is almost trivial, separating the cases  $p > 0$  and  $p = 0$ . ■

We can give an alternative version of Main Theorem 1.1, solely in terms of  $n$ ,  $s$ , and  $k$ , as follows.

**THEOREM 3.5.** *Following the notation and assumptions of Main Theorem 1.1, the number of connected components of  $S$  is also bounded above by  $4^{n-\frac{1}{2}}(s+1)^n(2(n+1)(s+1)+1)^{k+1}2^{k(k+1)/2}$ . ■*

The proof is very similar to that of corollary 3.2, save only that we substitute the polynomial system from the proof of Main Theorem 1.1 into the construction of  $\bar{F}_{\delta, \varepsilon}$ . In particular, we eventually obtain a system of  $n+1$  polynomials of degree  $2(s+1)$  in a total of  $k+1$  monomials, thus allowing yet another application of Khovanski's beautiful theorem on fewnomials.

#### 4. ALPHA THEORY AND PROVING MAIN THEOREMS 1.2 AND 1.3

The proof of Main Theorem 1.2 hinges on **gamma theory** [BCSS98], which gives useful criteria for when Newton's method converges quadratically. In particular, we will need the following elementary analytic lemma.

LEMMA 4.1. *For any monotonic function  $\phi : \mathbb{R} \rightarrow \mathbb{R}$ , let  $\gamma_\phi$  satisfy  $\sup_{k \geq 1} \left| \frac{\phi^{(k)}(x)}{k! \phi'(x)} \right|^{\frac{1}{k-1}} \leq \frac{\gamma_\phi}{x}$ . Then, for  $\phi(x) = x^r$ , we may take  $\gamma_\phi$  equal to  $\lceil |r| \rceil$ , 2 or 1, according as  $r \in (-\infty, -1) \cup (1, \infty)$ ,  $r \in (0, 1)$ , or  $r \in (-1, 0)$ . More generally, if  $\phi = \phi_1 + \phi_2$  with  $\phi_1$  and  $\phi_2$  both convex and either both increasing or both decreasing, then we can take  $\gamma_\phi = \max\{\gamma_{\phi_1}, \gamma_{\phi_2}\}$ . ■*

The first part is a simple exercise while the second part is a proposition from [Ye94].

We are now ready to sketch the proof of Main Theorem 1.2.

**Proof of Main Theorem 1.2:** We begin by changing our function  $f$  slightly. First let  $M$  be largest exponent occurring in the  $k$ -sum  $f$  and let  $m$  be the smallest real number so that  $x^m$  is a monomial term of  $f$  with **positive** coefficient. (We assume, by multiplying by  $-1$  if necessary, that the leading coefficient of  $f$  is positive.) By dividing out by  $x^m$  we may assume that  $m = 0$ . Via the change of variables  $x = y^{1/M}$ , we may further assume that  $M = 1$ . In particular, we now obtain that  $f$  is a sum of two increasing convex functions: one a positive linear combination of powers of  $x$  (with exponents in  $(0, 1]$ ), the other a negative linear combination of powers of  $x$  (with exponents in  $(-\infty, 0)$ ).

By our preceding lemma, we may take  $\gamma_f = d$  (the degree of  $f$ ) since  $d$  is no smaller than the degree of our original  $f$ . We now invoke the hybrid algorithm from [Ye94, Theorem 3]: This algorithm allows us to  $\varepsilon$ -approximate the real roots of  $f$  in  $(0, R)$  using  $\mathcal{O}(\log \gamma_f + \log \log \frac{R}{\varepsilon}) = \mathcal{O}(\log d + \log \log \frac{R}{\varepsilon})$  function evaluations and arithmetic operations. To conclude the first part of this main theorem, inverting the change of variables we made requires another  $\mathcal{O}(\log d + \log \log \frac{R}{\varepsilon})$  operations via the same algorithm (since taking  $n^{\text{th}}$  roots is the same as solving an exponential 2-sum). However, we may have decreased the accuracy of our  $\varepsilon$ -approximation. So we just begin by solving to accuracy  $\min\{\varepsilon^{M-m}, \varepsilon\}$  instead to obtain the first part of our main theorem. (Note also that evaluating  $f$  requires  $k$  uses of our oracle.)

To obtain the second part of our theorem, we simply use the same algorithm without the oracle. This simply introduces another factor of  $\log d$  since monomials can now be evaluated by the usual repeated squaring trick. ■

Main Theorem 1.3 only needs a special case of Main Theorem 1.2. In fact, [Ye94] contains a slightly modified algorithm for the binomial case

with an even better complexity bound of  $\mathcal{O}(\log d \log \log \frac{R}{\varepsilon})$ , which we will use below. However, we will also require some refined quantitative facts about the Smith normal form of a matrix.

LEMMA 4.2. [Ili89] *Let  $A = [a_{ij}]$  be any  $n \times n$  matrix with entries only in  $\mathbb{Z}$  and define  $h_A$  to be  $\log(2n + \max |a_{ij}|)$ . Then, within  $\mathcal{O}^*((n + h_A)^{6.375})$  bit operations, one can find matrices  $U, D, V$  with the following properties:*

1.  $U$  and  $V$  both have determinant  $\pm 1$  and entries only in  $\mathbb{Z}$ .
2.  $D$  is diagonal and has entries only in  $\mathbb{Z}$ .
3.  $UAV = D$
4.  $\det A$  is the product of the diagonal elements of  $D$  and  $h_U, h_V = \mathcal{O}(n^3(h_A + \log n)^2)$ .

**Proof of Main Theorem 1.3:** We begin by immediately applying the Smith normal form to our matrix  $[d_{ij}]$ . (This accounts for the bit operation count.) Clearly then, we have reduced to the case of  $n$  binomials of the form  $x_1^{d_1} - \gamma_1, \dots, x_n^{d_n} - \gamma_n$ . The real roots of this polynomial system can then be  $\varepsilon$ -approximated by  $n$  applications of Main Theorem 1.2. Since  $\sum_i \log d_i = \log \prod_i d_i = |\det[d_{ij}]|$ , this accounts for almost all of the second bound.

To conclude, note that we must still invert our change of variables. By lemma 4.2, computing this monomial map is almost the final contribution to our second complexity bound. The only missing part is the fact that we may have needed more accuracy at the beginning of our algorithm. Lemma 4.2 also tells us how much more accuracy we need, thus finally accounting for all of our second complexity bound. ■

## 5. SMALE'S THEOREM AND MAIN THEOREM 1.4

We begin with the following result of Plaisted.

PLAISTED'S THEOREM 5.. [Pla84] *Deciding if an input polynomial  $f \in \mathbb{Z}[x_1]$  (coefficients) vanishes at a  $d^{\text{th}}$  root of unity, where  $d = \deg(f)$ , is  $\mathbf{NP}$ -hard.* ■

In the above (and in what follows)  $f$  is given in the sparse encoding, so coefficients **and** exponents are measured by bit-length.

The following unpublished result of Steve Smale gives an intriguing extension of Plaisted's result via computations over new rings.

SMALE'S THEOREM 5.. *Suppose we can decide, within polynomial time relative to the BSS model over  $\mathbb{C}$ , if an input polynomial  $f \in \mathbb{C}[x_1]$  vanishes at a  $d^{\text{th}}$  root of unity, where  $d = \deg(f)$ . Then  $\mathbf{NP} \subseteq \mathbf{BPP}$ .*



**Proof:** Given any complexity class  $\mathcal{C}$  over the Turing model, consider its extension  $\mathcal{C}_{\mathbb{C}}$  to the BSS model over  $\mathbb{C}$ . It is then a simple fact that  $\mathcal{C}$  is contained in the **Boolean part** of  $\mathcal{C}_{\mathbb{C}}$ ,  $\mathbf{BP}(\mathcal{C}_{\mathbb{C}})$  [CKKLW95]. However, we will make use of an inclusion going the opposite way:  $\mathbf{BP}(\mathcal{C}_{\mathbb{C}}) \subseteq \mathcal{C}^{\mathbf{BPP}}$  [CKKLW95]. Applying this to the problem at hand, we thus see that the hypothesis of Smale's theorem, thanks to Plaisted's Theorem, implies that  $\mathbf{NP} \subseteq \mathbf{BP}(\mathcal{P}_{\mathbb{C}}) = \mathbf{P}^{\mathbf{BPP}} = \mathbf{BPP}$ . So we are done. ■

Our final main theorem then follows from some simple reductions to problem (1) from the statement.

**Proof of Main Theorem 1.4:** First note that the assertion concerning problem (1) follows immediately from Smale's Theorem and Plaisted's Theorem. It thus suffices to successively reduce (1) to special cases of all the other problems.

The assertion for (2) is then clear, since via the special case  $g(x) = x^d - 1$ , any polynomial time algorithm for (2) would give a polynomial time algorithm for (1).

On the other hand, a polynomial time algorithm for problem (5) would imply a polynomial time algorithm for problem (2). This is because problem (2) is essentially the decision problem of whether the **sparse resultant** of  $f$  and  $g$  [GKZ94] is zero. Via the **Cayley trick** [GKZ94], the  $\mathcal{A}$ -discriminant for  $\mathcal{A} = P \cup (Q \times \hat{e}_2)$  (where  $P$  and  $Q$  are respectively the **supports**<sup>8</sup> of  $f$  and  $g$ ) is exactly the sparse resultant of  $f$  and  $g$ . So this portion is done.

Note also that (3) is just a reformulation of (5).

As for (4), via the Jacobian criterion for singularities [Mum95] applied to the real and imaginary parts of the input to (3), a polynomial time algorithm for (4) (using the straight-line program encoding for the input) would immediately imply a polynomial time algorithm for (3) (using the straight-line program encoding for the input). Such an algorithm would then immediately be a polynomial time algorithm for (3) with inputs given in the sparse encoding. ■

## 6. ACKNOWLEDGEMENTS

The author thanks Felipe Cucker, Askold Khovanski, and Steve Smale for some very useful discussions. In particular, he thanks Felipe Cucker for pointing out an elegant proof of Smale's Theorem. Special thanks also go to Francois Loeser for pointing out the excellent paper [BLR91].

---

<sup>8</sup>The support of a polynomial is simply the set of its exponent vectors (fixing an ordering on the variables).

## REFERENCES

- Bar56. Barna, Bela, “Über die Divergenzpunkte des Newtonschen Verfahrens zur Bestimmung von Wurzeln Algebraischer Gleichungen,” Publ. Math. Debrecen, vol. 4, pp. 384–397 (1956).
- Bas96. Basu, Saugata, “On Bounding the Betti Numbers and Computing the Euler Characteristic of Semi-Algebraic Sets,” Proceedings of the Twenty-eighth Annual ACM STOC (Philadelphia, PA, 1996), pp. 408–417, ACM, New York.
- BPR96. Basu, S., Pollack, R., Roy, M.-F., “On the Combinatorial and Algebraic Complexity of Quantifier Elimination,” Journal of the ACM, Vol. 43, No. 6, November 1996, pp. 1002–1045.
- BLR91. Benedetti, R., Loeser, F., Risler, J. J., “Bounding the Number of Connected Components of a Real Algebraic Set,” Discrete and Computational Geometry, 6:191–209 (1991).
- BKK76. Bernshtein, D. N., Kushnirenko, A. G., and Khovanski, A. G., “Newton Polyhedra,” Uspehi Mat. Nauk 31 (1976), no. 3(189), pp. 201–202.
- BCSS98. Blum, L., Cucker, F., Shub, M., Smale, S., *Complexity and Real Computation*, Springer-Verlag, 1998.
- BZ88. Burago, Yu. D. and Zalgaller, V. A., *Geometric Inequalities*, Grundlehren der mathematischen Wissenschaften 285, Springer-Verlag (1988).
- CKKLW95. Cucker, F., Karpinski, M., Koiran, P., and Lickteig, T., and Werther, K., “On Real Turing Machines that Toss Coins,” Proceedings of the 27<sup>th</sup> STOC, pp. 335–342, ACM Press, 1995.
- CKS99. Cucker, F., Koiran, P., and Smale, S., “A Polynomial Time Algorithm for Diophantine Equations in One Variable,” Journal of Symbolic Computation (1999) **27**, pp. 21–29.
- DK86. Danilov, V. I. and Khovanski, A. G., “Newton Polyhedra and an Algorithm for Calculating Hodge-Deligne Numbers,” Math. USSR-Izv. 29 (1987), no. 2, pp. 279–298.
- DL79. Dobkin, David and Lipton, Richard, “On the Complexity of Computations Under Varying Sets of Primitives,” J. of Computer and System Sciences 18, pp. 86–91, 1979.
- DGH98. Dyer, M., Gritzmann, P., and Hufnagel, A., “On the Complexity of Computing Mixed Volumes,” SIAM J. Comput. **27** (1998), no. 2, pp. 356–400.
- GKZ94. Gel’fand, I. M., Kapranov, M. M., and Zelevinsky, A. V., *Discriminants, Resultants and Multidimensional Determinants*, Birkhäuser, Boston, 1994.
- GLS99. Giusti, M., Lecerf, G., and Salvy, B., “A Gröbner-Free Alternative to Polynomial System Solving,” preprint, TERA, 1999.
- GK94. Gritzmann, Peter and Klee, Victor, “On the Complexity of Some Basic Problems in Computational Convexity II: Volume and Mixed Volumes,” Polytopes: Abstract, Convex, and Computational (Scarborough, ON, 1993), pp. 373–466, NATO Adv. Sci. Inst. Ser. C Math. Phys. Sci., 440, Kluwer Acad. Publ., Dordrecht, 1994.
- GHMP95. Giusti, M., Heintz, J., Morais, J. E., Pardo, L. M., “When Polynomial Equation Systems can be ‘Solved’ Fast?,” Applied Algebra, Algebraic Algorithms and Error-Correcting Codes (Paris, 1995), 205–231, Lecture Notes in Comput. Sci. 948, Springer, Berlin, 1995.
- Hir94. Hirsch, Morris, *Differential Topology*, corrected reprint of the 1976 original, Graduate Texts in Mathematics, 33, Springer-Verlag, New York, 1994.

- Ili89. Iliopoulos, Costas S., “Worst Case Complexity Bounds on Algorithms for Computing the Canonical Structure of Finite Abelian Groups and the Hermite and Smith Normal Forms of an Integer Matrix,” *SIAM Journal on Computing*, 18 (1989), no. 4, pp. 658–669.
- Kho91. Khovanski, Askold, *Fewnomials*, AMS Press, Providence, Rhode Island, 1991.
- Koi96. Koiran, Pascal, “Hilbert’s Nullstellensatz is in the Polynomial Hierarchy,” DI-MACS Technical Report 96-27, July 1996. (**Note:** This preprint considerably improves the published version which appeared in *Journal of Complexity* in 1996.)
- Koi99. \_\_\_\_\_, “The Real Dimension Problem is  $\mathbf{NP}_{\mathbb{R}}$ -Complete,” LIP Research Report 97-36 (ENS Lyon), to appear in *Journal of Complexity*.
- Len98. Lenstra, Hendrik W., “Finding Small Degree Factors of Lacunary Polynomials,” *Number Theory in Progress*, proceedings of a meeting in honor of the 70<sup>th</sup> birthday of Andrej Schnizel, W. de Gruyter, to appear.
- Mil64. Milnor, John, “On the Betti Numbers of Real Varieties,” *Proceedings of the Amer. Math. Soc.* 15, pp. 275–280, 1964.
- MP98. Mourrain, Bernard and Pan, Victor Y. “Asymptotic Acceleration of Solving Multivariate Polynomial Systems of Equations,” *Proc. ACM STOC* 1998.
- Mum95. Mumford, David, *Algebraic Geometry I: Complex Projective Varieties*, Reprint of the 1976 edition, *Classics in Mathematics*, Springer-Verlag, Berlin, 1995.
- NR96. Neff, C. Andrew and Reif, John, “An Efficient Algorithm for the Complex Roots Problem,” *Journal of Complexity* 12 (1996), no. 2, pp. 81–115.
- OP49. Oleinik, O. and Petrovski, I., “On the Topology of Real Algebraic Hypersurfaces,” *Izv. Akad. Nauk SSSR* 13, pp. 389–402, 1949.
- Pap95. Papadimitriou, Christos H., *Computational Complexity*, Addison-Wesley, 1995.
- Pla84. Plaisted, David A., “New NP-Hard and NP-Complete Polynomial and Integer Divisibility Problems,” *Theoret. Comput. Sci.* 31 (1984), no. 1–2, 125–138.
- Ric93. Richardson, D., “Finding the Number of Distinct Real Roots of Sparse Polynomials of the Form  $p(x, x^n)$ ,” *Computational Algebraic Geometry (Nice, 1992)*, 225–233, *Progr. Math.*, 109, Birkhäuser, Boston, MA, 1993.
- Roj97. Rojas, J. Maurice, “Toric Laminations, Sparse Generalized Characteristic Polynomials, and a Refinement of Hilbert’s Tenth Problem,” *Foundations of Computational Mathematics (Rio de Janeiro, January 1997)*, pp. 369–381, Felipe Cucker and Mike Shub (eds.), Springer-Verlag (1997).
- Roj98. \_\_\_\_\_, “Intrinsic Near Quadratic Complexity Bounds for Real Multivariate Root Counting,” *Proceedings of the Sixth Annual European Symposium on Algorithms, Lecture Notes on Computer Science*, pp. 127–138, vol. 1461, Springer-Verlag (1998).
- Roj99a. \_\_\_\_\_, “On the Complexity of Diophantine Geometry in Low Dimensions,” *Proceedings of the 31<sup>st</sup> Annual ACM STOC (May 1-4, 1999, Atlanta, Georgia)*, pp. 527–536, ACM Press, 1999.
- Roj99b. \_\_\_\_\_, “Solving Degenerate Sparse Polynomial Systems Faster,” *Journal of Symbolic Computation*, vol. 28 (special issue on elimination theory), no. 1/2, July and August 1999, pp. 155–186.
- RY99. Rojas, J. M. and Ye, Yinyu, “Solving Real Fewnomials in Logarithmic Time,” submitted for publication, 1999.
- Smi61. Smith, H. J. S., “On Systems of Integer Equations and Congruences,” *Philos. Trans.* 151, pp. 293–326 (1861).

- SY82. Steele, J. and Yao, A., "*Lower Bounds for Algebraic Decision Trees*," J. of Algorithms 3, pp. 1–8, 1982.
- Stu93. Sturmfels, Bernd, "*Sparse Elimination Theory*," In D. Eisenbud and L. Robbiano, editors, Proc. Computat. Algebraic Geom. and Commut. Algebra 1991, pages 377–396, Cortona, Italy, 1993, Cambridge Univ. Press.
- Tho65. Thom, René, "*Sur l'homologie des variétés algébriques réelles*," In S. Cairns (Ed.), Differential and Combinatorial Topology, Princeton University Press, 1965.
- Ye94. Ye, Yinyu, "*Combining Binary Search and Newton's Method to Compute Real Roots for a Class of Real Functions*," J. Complexity 10 (1994), no. 3, 271–280.